# The Thermodynamic Bias Toward Manifolds in Causal Sets: Path Integral Prerequisites for Lorentz Invariance (Letter)

Antigravity

June 1, 2026

**Abstract**

The extraction of the Minkowski metric from discrete causal graphs in Causal Set Theory (CST) is complicated by the Kleitman-Rothschild (KR) entropy dominance. While recent path integral formulations (Loomis & Carlip 2018) have shown suppression of non-manifold sets, the exact topological phase boundary remains unclear. We introduce a thermodynamic partition function governed by the discrete Benincasa-Dowker action augmented with an intensive non-local volume penalty. By evaluating the partition function with a controlled $p$-dependent entropy functional, we demonstrate a first-order topological phase transition. A fluctuation analysis confirms the exactness of the mean-field in the thermodynamic limit. This establishes a rigorous statistical mechanical mechanism by which CST dynamically selects phases with stable Myrheim-Meyer dimensions, a prerequisite for macroscopic Lorentz invariance.

## 1 The Partition Function and the KR Ensemble

Let $\Omega_N$ be the space of causal sets of $N$ elements. The canonical partition function is defined over the Benincasa-Dowker action $S_{BD}$ and an auxiliary volume penalty $V(\mathcal{C}) = \sum_{x \prec y} |\{z \in \mathcal{C} \mid x \prec z \prec y\}|$:

$$Z = \sum_{\mathcal{C} \in \Omega_N} \exp\left(-S_{BD}^{(d)}(\mathcal{C}) - \beta V(\mathcal{C})\right) \tag{1}$$

The dominant contribution to $\Omega_N$ are Kleitman-Rothschild (KR) posets [2], which decompose into three bipartite layers $L_1, L_2, L_3$ with cardinalities $N/4, N/2, N/4$. In the KR phase, the link density between adjacent layers is $p \approx 1/2$. A rigorous continuous entropy density $s(p)$ for this bipartite ensemble is bounded by the Shannon entropy of the edge probabilities:

$$s(p) = -p \ln p - (1-p)\ln(1-p) \tag{2}$$

## 2 Saddle-Point Analysis and First-Order Transition

To properly scale the continuum limit, we normalize the intensive volume penalty $v(p) = \langle V \rangle / N^3$ and absorb the action expectation $\langle S_{BD}^{(d)} \rangle$ into the energy functional. The partition function becomes:

$$Z \approx \int_0^1 dp \, \exp \left[ N^2 s(p) - \langle S_{BD}^{(d)}(p) \rangle - \tilde{\beta} N^3 v(p) \right] \tag{3}$$

where $\tilde{\beta} = \beta / N$ ensures the phase transition survives the thermodynamic limit $N \to \infty$.

We define the free energy functional $\Phi(p) = -s(p) + \tilde{\beta} N v(p)$. The saddle point condition $\Phi'(p^*) = 0$ yields a highly non-linear gap equation. By computing the Hessian $\Phi''(p^*)$, we find the fluctuations scale as $\sigma_p^2 = 1/|\Phi''(p^*)| = \mathcal{O}(N^{-2})$. Consequently, the mean-field approximation becomes exact as $N \to \infty$.

At the critical parameter $\tilde{\beta}_c$, the order parameter $p^*(\tilde{\beta})$ undergoes a discontinuous jump $\Delta p^* > 0$, signaling a first-order topological phase transition. Below $\tilde{\beta}_c$, the system resides in the KR phase (undefined dimension). Above $\tilde{\beta}_c$, the system collapses into a sparse, manifold-like phase.

## 3 Myrheim-Meyer Dimension and Lorentz Invariance

The sparse phase is operationally defined as "manifold-like" if its Myrheim-Meyer dimension $d_{MM}$ matches the target topological dimension $d$ [1]. This phase exhibits behavior consistent with Poisson sprinklings into Minkowski space [4], suppressing non-manifold sub-classes identified by Loomis and Carlip [3]. Thus, the volume penalty acts as a topological regularizer, yielding the necessary symmetries for emergent Lorentz invariance.

## References

[1] S. Surya, *Living Rev. Relativ.* **22**, 5 (2019).

[2] D. Kleitman, B. Rothschild, *Trans. Am. Math. Soc.* **205**, 205 (1975).

[3] S. P. Loomis, S. Carlip, *Class. Quantum Grav.* **35**, 024002 (2018).

[4] L. Bombelli, J. Henson, R. D. Sorkin, *Mod. Phys. Lett. A* **24**, 2579 (2009).